PART 1: QUESTIONS

Name:______________________________ **Age:**_____ **Id:**______________ **Course:**__________

Functions - Exam 2 **Lessons: 13 - 15**

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. A quadratic function is:

I. $y = ax^2 + bx + c, a \neq 0$
II. $y = (x - x_1)(x - x_2)$, where and x_1, x_2 are the roots.
III. $y = ax^2 + bx + c$, where a, b, and $c \in \mathbb{R}$.

a) Only I is correct.
b) Only II is correct.
c) Only III is correct.
d) I, II, and III are correct.
e) None of the above.

Solution: a

Only I is true.

By definition, a quadratic function is any function in the form $y = ax^2 + bx + c, a \neq 0$.
The quadratic function also can be written by the following factor formula:
$y = a(x - x_1)(x - x_2)$, where $a \neq 0$ and x_1, x_2 are the roots.

2. If x_1 and x_2 are the roots of a quadratic function $y = ax^2 + bx + c, a \neq 0$ then:

I. $x_1 + x_2 = \dfrac{b}{a}$

II. $x_1 . x_2 = \dfrac{c}{a}$

III. $x_1 = \dfrac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$.

a) Only I and II are correct.
b) Only I and III are correct.
c) Only II and III are correct.
d) I, II, and III are correct.
e) None of the above.

Solution: c

I. False.

The sum of the roots is $x_1 + x_2 = -\dfrac{b}{a}$.

II. True.

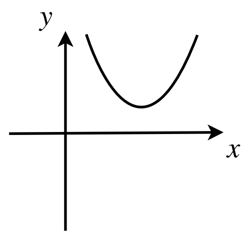
The product of the roots is $x_1 . x_2 = \dfrac{c}{a}$.

III. True.

The roots x_1 and x_2 of a quadratic function are calculated by the following quadratic formula:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac.$$

3. Given the graph of the quadratic function $y = ax^2 + bx + c, a \neq 0$ such that x_1 and x_2 are its roots:



Then:

a) $a > 0$ and $\Delta > 0$
b) $a > 0$ and $\Delta = 0$
c) $a > 0$ and $\Delta < 0$
d) $a < 0$ and $\Delta > 0$
e) $a < 0$ and $\Delta = 0$.

Solution: c

The concavity is upward then $a > 0$. There are no root (x_1 and x_2) then $\Delta < 0$.

4. Let $y = ax^2 + bx + c, a \neq 0$ be a quadratic function with vertex $V(x_v, y_v)$.

I. $x_v = \dfrac{\sqrt{x_1 . x_2}}{2}$ and $y_v = a(x_v^2) + b(x_v) + c$, where x_1 and x_2 are the roots.

II. $x_v = \dfrac{b}{2a}$ and $y_v = \dfrac{-\Delta}{4a}$

III. $x_v = \dfrac{x_1 + x_2}{2}$ and $y_v = a(x_v^2) + b(x_v) + c$, where x_1 and x_2 are the roots.

a) Only I is correct.
b) Only II is correct.
c) Only III is correct.
d) Only I and III are correct.
e) None of the above.

Solution: c

The vertex of the parabola is $V\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$.

By symmetry, $x_v = \dfrac{x_1 + x_2}{2}$ and
$y_v = a(x_v^2) + b(x_v) + c$, where x_1 and x_2 are the roots.

Thus, only III is correct.

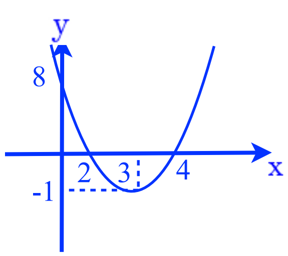
5. The value of m such that the quadratic function $y = mx^2 + 2x + 2$ has two distinct roots is:

a) $m < \frac{1}{16}$
b) $m < \frac{1}{8}$
c) $m < \frac{1}{4}$
d) $m < \frac{1}{2}$
e) None of the above.

Solution: d

The quadratic function has two distinct roots if $\Delta > 0$.
$y = mx^2 + 2x + 2$ ($a = m$, $b = 2$, and $c = 2$)
$\Delta = b^2 - 4ac$
$\Delta = (2)^2 - 4(m)(2) > 0$
$4 - 8m > 0$
$8m < 4$
$m < \frac{1}{2}$.

6. The quadratic function of the following graph is:



a) $y = x^2 - 6x + 8$
b) $y = x^2 - 4$
c) $y = -x^2 + 2x + 3$
d) $y = x^2 - x - 12$
e) None of the above.

Solution: a

Let $x_1 = 2$ and $x_2 = 4$ be the roots of y. Then:
$y = a(x - x_1)(x - x_2) \Rightarrow y = a(x - 2)(x - 4)$
Since $A(0,8)$ belong to the graph, we have:
$8 = a(0 - 2)(0 - 4) \Rightarrow 8 = 8a \Rightarrow a = 1$.
Then, $y = a(x - 2)(x - 4) \Rightarrow y = 1(x - 2)(x - 4)$
Thus, $y = x^2 - 6x + 8$.

Note: By symmetry, $x_v = \frac{x_1 + x_2}{2} \Rightarrow x_v = \frac{2 + 4}{2} \Rightarrow$
$x_v = 3$

$y = x^2 - 6x + 8 \Rightarrow y_v = (x_v^2) - 6(x_v) + 8 \Rightarrow$
$y_v = (3^2) - 6(3) + 8 \Rightarrow y_v = -1$.

7. Find the minimum value of the following quadratic function:

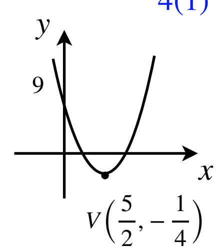
$y = x^2 - 5x + 6$

a) -9 b) -4 c) -1 d) 1 e) None of the Above.

Solution: e

$y = x^2 - 5x + 6$

$x_v = \frac{-b}{2a} \Rightarrow x_v = \frac{-(-5)}{2(1)} \Rightarrow x_v = \frac{5}{2}$
$y_v = \frac{-\Delta}{4a} \Rightarrow y_v = \frac{-(b^2 - 4ac)}{4a} \Rightarrow$
$y_v = \frac{-[(-5)^2 - 4(1)(6)]}{4(1)} \Rightarrow y_v = -\frac{1}{4}$

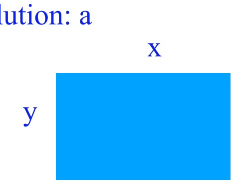


The minimum value of the quadratic function is $-\frac{1}{4}$.

8. The perimeter of a rectangle is 8 ft. The maximum area of the rectangle is:

a) 4 ft^2
b) 16 ft^2
c) 36 ft^2
d) 625 ft^2
e) None of the above.

Solution: a



$p = 2x + 2y = 8 \Rightarrow x + y = 4 \Rightarrow y = 4 - x$
$A = xy \Rightarrow A = x(4 - x) \Rightarrow A = -x^2 + 4x$
Note: Since $a < 0$ then we have a maximum.

$$x_v = \frac{-b}{2a} \Rightarrow x_v = \frac{-(4)}{2(-1)} \Rightarrow x_v = 2$$
$$y_v = \frac{-\Delta}{4a} \Rightarrow y_v = \frac{-(b^2 - 4ac)}{4a} \Rightarrow$$
$$y_v = \frac{-[(4)^2 - 4(-1)(0)]}{4(-1)} \Rightarrow y_v = 4$$

Thus, the maximum area is 4 ft^2.

9. Find the formula for the revenue function if the price-demand function of a product is $p = 80 - 2x$, where x is the number of items sold and the price is in dollars. How many items should be sold in order to maximize the revenue? What is the maximum revenue?

a) 2 items and \$40
b) 4 items and \$32
c) 9 items and \$243
d) 20 items and \$800
e) None of the above.

Solution: d

$p = 80 - 2x$; where x is the number of items sold.
$R(x) = p\,.\,x \Rightarrow R(x) = (80 - 2x)x \Rightarrow R(x) = -2x^2 + 80x$

$$x_v = \frac{-b}{2a} \Rightarrow x_v = \frac{-(80)}{2(-2)} \Rightarrow x_v = 20 \text{ items.}$$
$$y_v = \frac{-\Delta}{4a} \Rightarrow y_v = \frac{-(b^2 - 4ac)}{4a} \Rightarrow$$
$$y_v = \frac{-[(80)^2 - 4(-2)(0)]}{4(-2)} \Rightarrow y_v = \$800.$$

10. Given:

I. When the polynomial $p(x)$ is divided by $x - a$, the remainder is $p(a)$.

II. If $p(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$ then there exist unique polynomials $q(x)$ and $r(x)$ such that: $p(x) = d(x)\,.\,q(x) + r(x)$; where the degree of $r(x)$ is less than the degree of $d(x)$.

III. A polynomial function is any function in the form: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$; $a_n \neq 0$.

Then:

a) Only I and II are correct.
b) I, II, and III are correct.
c) Only II and III are correct.
d) I, II, and III are incorrect.
e) None of the above.

Solution: c

I. True. The remainder theorem.
II. True. Definition of polynomial division.
III. True. Definition of polynomial.

11. Given:

I. $p(x) = 3x^5 + 1$
II. $p(x) = \sqrt{-1}x + 2$
III. $p(x) = 1$

Then:

a) I, II, and III are polynomials.
b) Only I and III are polynomials.
c) Only II and II are polynomials.
d) Only I and III are polynomials.
e) None of the above.

Solution: d

A polynomial function has the form:

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$ and n is a nonnegative integer.

Thus, only I and III are polynomials.

12. Given $p(x) = 2x^2 - 1$ and $d(x) = x + 1$, then:

I. $p(x) - d(x) = 2x^2 - x - 2$
II. $p(x)\,.\,d(x) = 2x^3 + 2x^2 - x - 1$
III. In the division $\dfrac{p(x)}{d(x)}$, the quotient is $q(x) = 2x - 2$ and the remainder is $r(x) = -3$.

a) Only I and II are correct
b) Only I and III are correct
c) Only II and III are correct
d) I, II, and III are correct
e) None of the above.

Solution: a

I. True. $p(x) - d(x) = 2x^2 - 1 - (x + 1) = 2x^2 - 1 - x - 1 = 2x^2 - x - 2$

II. True. $p(x)\,.\,d(x) = (2x^2 - 1)(x + 1) = 2x^3 + 2x^2 - x - 1$

III. false. $\dfrac{p(x)}{d(x)} = ?$

$$\begin{array}{r|l} & 2x - 2 \\ x + 1 & 2x^2 + 0x - 1 \\ & \underline{2x^2 + 2x} \\ & \quad -2x - 1 \\ & \quad \underline{-2x - 2} \\ & \qquad\quad 1 \end{array}$$

Then, the quotient is $q(x) = 2x - 2$ and the remainder is $r(x) = 1$.

Thus, only I and II are correct.

13. The remainder of $p(x) = x^3 - x^2 + 1$ by $d(x) = x + 1$ is:

a) -1 b) 2 c) 3 d) 4 e) None of the above.

Solution: a

The remainder theorem: Let $p(x)$ be any polynomial of degree greater than or equal to 1 and let a be any real number. If $p(x)$ is divided by the polynomial $(x - a)$, then the remainder is $p(a)$.
Then,
$a = -1$ and the remainder is $r = p(-1)$.
$r = (-1)^3 - (-1)^2 + 1$
$r = -1$.

14. Let x_1, x_2, and x_3 be the roots of $p(x) = x^3 - 6x^2 + 11x - 6$. Given $x_1 = 1$ then $k = (x_2)^2 + (x_3)^2$ is:

a) $k = 5$
b) $k = 10$
c) $k = 13$
d) $k = 25$
e) None of the above.

Solution: c

$x_1 = 1$

By the synthetic method:

1	1	-6	11	-6
	1	-5	6	0

$x^2 - 5x + 6 = 0$

$S = \dfrac{-b}{a} \Rightarrow S = \dfrac{-(-5)}{1} \Rightarrow S = 5$

$P = \dfrac{c}{a} \Rightarrow P = \dfrac{6}{1} \Rightarrow P = 6$

+1 +6
+2 +3 $x_2 = 2$ and $x_3 = 3$.

Thus, $k = (x_2)^2 + (x_3)^2 = 2^2 + 3^2 = 13$.

15. Let $q(x)$ and $r(x)$ be the quotient and remainder by the division of $p(x) = x^3 - 1$ by $d(x) = x^2 - 2$. Then $q(x) + r(x)$ is:

a) $2x - 1$
b) $x^2 - 1$
c) $-x^2 + 1$
d) $x^2 - x + 1$
e) None of the above.

Solution: e

Using the long method for polynomials, we have:

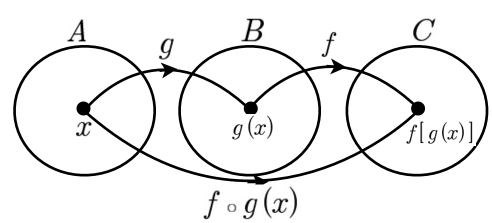
$$\begin{array}{r|l} & x \\ x^2 - 2 & x^3 + 0x^2 + 0x - 1 \\ & \underline{x^3 \qquad\quad -2x \quad} \\ & \qquad\qquad 2x - 1 \end{array}$$

Then, $q(x) = x$ and $r(x) = 2x - 1$.
Thus, $q(x) + r(x) = (x) + (2x - 1) = 3x - 1$.

16. Let $g : A \to B$ and $f : A \to B$ be functions. The composition of function $fog(x) = f[g(x)]$ exists if:
Notation: Im : Image and D : Domain.

a) $Im_g = Im_f$
b) $Im_g = D_f$
c) $D_g = D_f$
d) $Im_f = D_g$
e) None of the above.

Solution: b



The composition of function $fog(x) = f[g(x)]$ exists if $Im_g = D_f$.

17. Given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x - 3$ and $g(x) = 3x + 1$. Then $f[g(1)]$ is:

a) 1 b) 3 c) 14 d) 16 e) None of the above.

Solution: b

Let $f(x) = x - 1$ and $g(x) = 3x + 1$. Then:

$$\begin{aligned} f[g(x)] &= g(x) - 1 \\ &= (3x + 1) - 1 \\ &= 3x \end{aligned}$$

Thus, $f[g(1)] = 3(1) = 3$.

18. Given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 5x - 1$ and $f[g(x)] = 2 - 3x$. Then:

a) $g(x) = \dfrac{x}{2} + 1$
b) $g(x) = 2x - 8$
c) $g(x) = \dfrac{3x}{4} + \dfrac{3}{4}$
d) $g(x) = -\dfrac{3x}{5} + \dfrac{3}{5}$
e) None of the above.

Solution: d

Since $f(x) = 5x - 1$ then:

$$\begin{aligned} f[g(x)] &= 5[g(x)] - 1 \\ 2 - 3x &= 5g(x) - 1 \\ 5g(x) &= -3x + 3. \\ g(x) &= -\frac{3x}{5} + \frac{3}{5}. \end{aligned}$$

19. Given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x - 1$ and $g[f(x)] = 5 - 3x$. Then:

a) $g(x) = 2x + 7$
b) $g(x) = -3x + 2$
c) $g(x) = -2x + 13$
d) $g(x) = x - 4$
e) None of the above.

Solution: b

$f(x) = x - 1 \Rightarrow x = f(x) + 1$
$g[f(x)] = 5 - 3x$
$g[f(x)] = 5 - 3[f(x) + 1]$
$g[f(x)] = 5 - 3f(x) - 3$
$g[f(x)] = -3f(x) + 2$
Thus, $g(x) = -3x + 2$.

20. If $g(x) = \sqrt[5]{x + 2}$ and $h(x) = x^5 + 3$. Then:

a) $h[g(x)] = x$
b) $h[g(x)] = x + 5$
c) $h[g(x)] = x - 1$
d) $h[g(x)] = x - 2$
e) None of the above.

Solution: b

$h(x) = x^5 + 3$
$h[g(x)] = [g(x)]^5 + 3$
$h[g(x)] = [\sqrt[5]{x + 2}]^5 + 3$
$h[g(x)] = x + 2 + 3$
$h[g(x)] = x + 5$.

Name:______________________________ **Age:**_____ **Id:**_______________ **Course:**__________

PART 2: SOLUTIONS

Consulting

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

21. Graph $y = x^2 - 2x$.

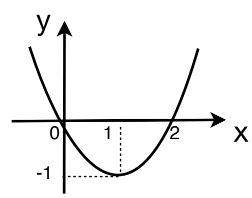
Solution:

$y = x^2 - 2x$

Find the roots:
$x^2 - 2x = 0$
$x(x - 2) = 0$
$x_1 = 0$ or $x_2 = 2$

Find the vertex:
$x_v = \dfrac{x_1 + x_2}{2} \Rightarrow x_v = \dfrac{0+2}{2} \Rightarrow x_v = 1$
$y_v = x_v^2 - 2x_v$
$y_v = (1)^2 - 2(1)$
$y_v = -1$, then $V(1, -1)$.

Thus, the graph is:



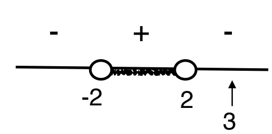
22. Calculate m such that the quadratic function $y = 4x^2 + 8x + m^2$ has two distinct real roots:

Solution:

$y = 4x^2 + 8x + m^2$ ($a = 4$, $b = 8$, and $c = m^2$)

If we have two distinct real roots then $\Delta > 0$.

$\Delta = b^2 - 4ac$
$\Delta = (8)^2 - 4(4)(m^2)$
$\Delta = 16(4 - m^2) > 0$
$(4 - m^2) > 0$
$(2 + m)(2 - m) > 0$



Thus, $S = \{m \in \mathbb{R} \,/\, -2 < m < 2\}$.

23. Given the equation $3x + y = 24$, find x and y such that the product $P = xy$ be a maximum.

Solution: $x_{max} = 4$ and $y_{max} = 12$.

$3x + y = 24 \Rightarrow y = 24 - 3x$

$P = xy \Rightarrow P = x(24 - 3x)$
$P = -3x^2 + 24x$ ($a = -3$, $b = 24$, and $c = 0$)
Note: $a < 0 \Rightarrow$ Downward parabola $\Rightarrow$ Maximum.

$x_{max} = \dfrac{-b}{2a} \Rightarrow x_{max} = \dfrac{-24}{2(-3)} \Rightarrow x_{max} = 4$
$y_{max} = 24 - 3(x_{max}) \Rightarrow y_{max} = 24 - 3(4) \Rightarrow$
$y_{max} = 12$.

24. Given $f(x) = \dfrac{5 - x}{\sqrt[7]{x - 7}}$. Find the domain of $f(x)$.

Solution: $D_f = \{x \in \mathbb{R} \,/\, x \neq 7\}$

$x - 7 \neq 0 \Rightarrow x \neq 7$

Thus, $D_f = \{x \in \mathbb{R} \,/\, x \neq 7\}$

25. Show me that Derivatives are easy. Let $f(x)$ be a polynomial such that $f(x) = x^n$. Then the derivative of $f(x)$ called $f'(x)$ is $f'(x) = nx^{n-1}$. Find the derivative of $f(x) = x^7$.

Solution: $f'(x) = 7x^6$

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$. For $n = 7$, we have:
$f(x) = x^7 \Rightarrow f'(x) = 7x^6$.
Solution: $f'(x) = 5x^4$