Name:______________________________ **Age:**_____ **Id:**______________ **Course:**_______

Integrals - Exam 1 **Lessons: 38 - 40** Consulting

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. Given:

I. Since the finding the integral of a function with respect to x means finding a area then the integral is always positive.

II. Sometimes, integrals can change the sign of the values for example when $f(x) < 0$ or when the direction of integration is towards the negative direction.

III. Since the finding the integral of a function with respect to x means always finding a area then the integral is never used to find volumes.

IV. Although finding the integral of a function with respect to x means finding the area to the x axis from the curve, an integral can be used to calculate displacement, area, volume, and other concepts that arise by combining infinitesimal data.

V. If F(x) is the antiderivative of function $f(x)$ then the integral of F(x) is f(x) + c.

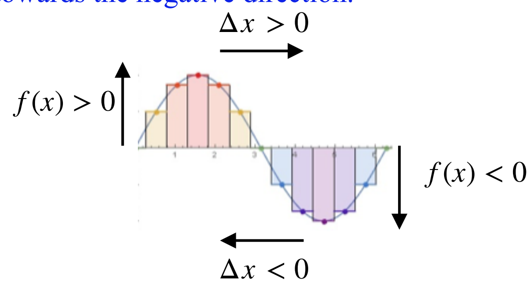
VI. An antiderivative of a function $f(x)$ is a function whose derivative is equal to $f(x)$. That is, if $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

a) Only I, III, V are correct.
b) Only II, IV and VI are correct.
c) Only II, III, and V are correct.
d) Only I, IV, and VI are correct.
e) None of the above.

Solution: b

The integral is usually called the anti-derivative, because integrating is the reverse process of differentiating.
An antiderivative of a function $f(x)$ is a function whose derivative is equal to $f(x)$. That is, if $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.
However, antiderivatives are not unique. A given function can have many antiderivatives. All antiderivatives of a given function differ by a constant. This allows us to write a general formula for any antiderivative by adding a constant. Antiderivatives are intimately connected with areas. However, integrals can take up negative values. In those cases, how can I assume they represent the area under the curve, since areas can never be negative? Integral is the sum of

rectangles with base Δx and heigh $f(x)$. Since Δx and $f(x)$ can be negative then the product $f(x)\Delta x$ can be negative as well. Thus integral can take negative values when $f(x) < 0$ or when the direction of integration is towards the negative direction.



2. Given:

I. Given a function $f(x)$ that is continuous on the interval $[a, b]$ and suppose that $F(x)$ is any anti-derivative for $f(x)$. Then the **Theorem Fundamental of Calculus** states:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

II. The definition of **indefinite integra**l is:
$\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$ and c is any constant.

III. The definition of **anti-derivative** of a function $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.

a) Only I and II are correct.
b) I, II, and III are correct.
c) Only I and III are correct.
d) Only II and III are correct.
e) None of the above.

Solution: b

All alternatives are correct.

3. Given:

I. $\int_a^a f(x)dx = 0$

II. $\int_a^b k\,.f(x)dx = k\int_a^b f(x)dx,\ k \in \mathbb{R}$

III. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

IV. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$

V. $\int_{\mathbf{b}}^{\mathbf{a}} f(x)dx = -\int_a^b f(x)dx$

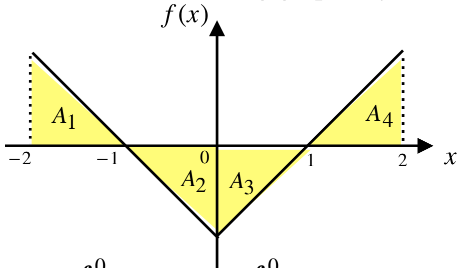
a) Only I is **incorrect**.
b) Only II is **incorrect**.
c) Only III is **incorrect**.
d) Only IV is **incorrect**.
e) Only V is **incorrect**..

Solution: d

IV is False because

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

4. Given the following graph of $f(x)$.



Then $\int_{-2}^{0} f(x)dx + \int_{2}^{0} f(x)dx$ is:

a) $-A_1 + A_2 + A_4 - A_3$
b) $A_1 - A_2 + A_4 - A_3$
c) $-A_1 + A_2 - A_4 + A_3$
d) $A_1 - A_2 - A_4 + A_3$
e) None of the above.

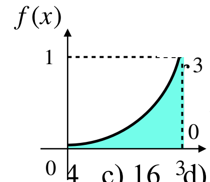
Solution: d

$$A = \int_{-2}^{0} f(x)dx + \int_{2}^{0} f(x)dx$$

$$A = \int_{-2}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx \int_{2}^{1} f(x)dx + \int_{1}^{0} f(x)dx$$

$$A = A_1 - A_2 - A_4 + A_3.$$

5. Given the graph of $f(x) = x^3$.



$f(x)\,dx.$

4 4 c) 16 d) $\frac{81}{4}$ e) None of the above.

Solution: d

$$\int_0^3 f(x)\,dx = \int_0^3 x^3\,dx = \frac{x^4}{4}\Big|_0^3 = \left(\frac{3^4}{4} - \frac{0^4}{4}\right) = \frac{81}{4}.$$

6. Find $I = \int 5x^4 + 3x^2\,dx$, where e is the Euler's number.

a) $x^7 + x^5 + c$
b) $x^6 + x^4 + c$
c) $x^5 + x^3 + c$
d) $x^4 + x^2 + c$
e) None of the above.

Solution: c

$$I = \int 5x^4 + 3x^2\,dx = \int 5x^4\,dx + \int 3x^2\,dx.$$
$$= x^5 + x^3 + c.$$

7. Find $I = \int_0^2 e^x dx$, where e is the Euler's number.

a) $e - 1$ b) $e^2 - 1$ c) $e^3 - 1$ d) $e^4 - 1$ e) $e^5 - 1$.

Solution: b

$$I = \int_0^2 e^x dx = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1.$$

8. Find $I = \int_0^{\frac{\pi}{4}} \cos x\,dx$.

a) $\frac{1}{2}$ b) $\frac{\sqrt{2}}{2}$ c) $\frac{\sqrt{3}}{2}$ d) 1 e) None of the above.

Solution: b

$$I = \int_0^{\frac{\pi}{4}} \cos x\,dx = \sin x\Big|_0^{\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) - \sin\left(0\right)$$

$$= \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.$$

9. Find $I = \int_0^{\pi} \sec^2 x\,dx.$

a) 0 b) $\frac{\sqrt{3}}{3}$ c) 1 d) $\sqrt{3}$ e) None of the above.

Solution: a

$$I = \int_0^{\pi} \sec^2 x\,dx = \tan x\Big|_0^{\pi} = \tan(\pi) - \tan(0)$$
$$= 0 - 0 = 0.$$

10. Find $I = \int_0^1 \frac{dx}{1+x^2}.$

a) 0 b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{3}$ e) None of the above.

Solution: e

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

11. Find $I = \int_e^{e^5} \frac{dx}{x}$, where e is the Euler's number.

a) 1 b) 2 c) 3 d) 4 e) None of the above.

Solution: d

$$I = \int_e^{e^5} \frac{dx}{x} = \ln|x|\Big|_e^{e^5} = \ln|e^5| - \ln|e|$$
$$= 5\ln|e| - \ln|e|$$
$$= 4\ln|e|$$
$$= 4.$$

12. Find $I = \int x^3 (e)^{x^4} dx$

a) $\frac{1}{2}(e)^{x^2} + c$

b) $\frac{1}{3}(e)^{x^3} + c$

c) $\frac{1}{4}(e)^{x^4} + c$

d) $\frac{1}{5}(e)^{x^5} + c$

e) None of the above.

Solution: c

Let $u = x^4 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow du = 4x^3 dx.$

$I = \int x^3 (e)^{x^4} dx \Rightarrow I = \frac{1}{4}\int (e)^{x^4} 4x^3 dx.$

Making the substitution,

$I = \frac{1}{4}\int e^u du = \frac{1}{4}e^u + c$

Thus, $I = \frac{1}{4}(e)^{x^4} + c.$

13. Find $I = \int_0^{\frac{\pi}{3}} \sin 3x\,dx$

a) $1 - \frac{\sqrt{3}}{2}$ b) $1 - \frac{\sqrt{2}}{2}$ c) $\frac{1}{2}$ d) 1 e) None of the above.

Solution: e

Let $u = 3x \Rightarrow \frac{du}{dx} = 3 \Rightarrow du = 3dx.$

$I = \int_0^{\frac{\pi}{3}} \sin 3x\,dx.$

Note: $u = 3(0) = 0$ and $u = 3(\frac{\pi}{3}) = \pi$

Making the substitution,

$$I = \int_0^{\pi} \sin u\,du = -\cos u\Big|_0^{\pi}$$
$$= -\left[\cos(\pi) - \cos(0)\right]$$
$$= -\left[-1 - 1\right]$$

Thus, $I = 2.$

14. Find $I = \int_{-2}^{0} x\sqrt{2x^2+1}\,dx$

a) $-\frac{13}{3}$

b) $-\left(\frac{\sqrt{3}}{2} - \frac{1}{6}\right)$

c) $\frac{\sqrt{3}}{2} - \frac{1}{6}$

d) $\frac{13}{3}$

e) None of the above.

Solution: a

Let $u = 2x^2 + 1 \Rightarrow \dfrac{du}{dx} = 4x \Rightarrow du = 4x\,dx.$

$$I = \frac{1}{4}\int_{-2}^{0} \sqrt{2x^2+1}\; 4x\,dx.$$

Note: $u = 2(-2)^2 + 1 = 9$ and $u = 2(0)^2 + 1 = 1$

Making the substitution,

$$I = \frac{1}{4}\int_{9}^{1} \sqrt{u}\,du$$

$$I = \frac{1}{4}\int_{9}^{1} u^{\frac{1}{2}}\,du$$

$$I = \frac{1}{4}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] = \frac{1}{6}\left[u^{\frac{3}{2}}\right]_9^1$$

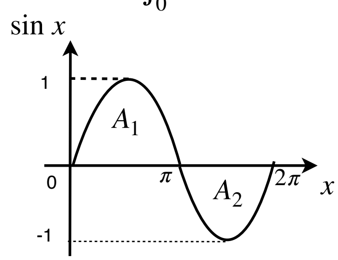
$$= \frac{1}{6}\left[(1)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$$

$$= \frac{1}{6}\left[1 - 27\right]$$

$$= -\frac{13}{3}$$

Thus, $I = -\dfrac{13}{3}$.

15. Find $I = \displaystyle\int_0^{2\pi} \sin x\,dx$



a) 0 b) 1 c) 2 d) 3 e) None of the above.

Solution: a

Since $A_1 = A_2 \Rightarrow I = A_1 - A_2 = 0.$

16. Find $I = \displaystyle\int \sin^2(2x)\cos(2x)\,dx$

a) $\dfrac{1}{6}\sin^3(2x) + c$

b) $\dfrac{1}{9}\sin^3(3x) + c$

c) $\dfrac{1}{12}\sin^3(4x) + c$

d) $\dfrac{1}{15}\sin^3(5x) + c$

e) None of the above.

Solution: a

Let

$u = \sin(2x) \Rightarrow \dfrac{du}{dx} = \cos(2x)(2) \Rightarrow du = 2\cos(2x)\,dx$

$I = \displaystyle\int \sin^2(2x)\cos(2x)\,dx \Rightarrow I = \frac{1}{2}\int \sin^2(2x)2\cos(2x)\,dx$

Making the substitution,

$$I = \frac{1}{2}\int u^2\,du$$

$$I = \frac{1}{2}\left(\frac{u^3}{3}\right) + c$$

$$I = \frac{1}{6}\sin^3(2x) + c.$$

17. Find $I = \displaystyle\int e^{(1+5\sin x)}\cos x\,dx$

a) $e^{(1+\sin x)} + c$

b) $\dfrac{1}{2}e^{(1+2\sin x)} + c$

c) $\dfrac{1}{3}e^{(1+3\sin x)} + c$

d) $\dfrac{1}{4}e^{(1+4\sin x)} + c$

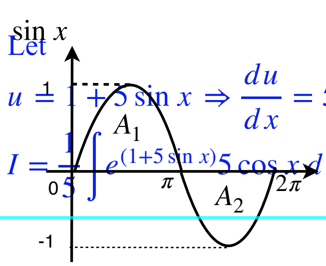
e) None of the above.

Solution: e

Let

$u = 1 + 5\sin x \Rightarrow \dfrac{du}{dx} = 5\cos x \Rightarrow du = 5\cos x\,dx\,.$

$I = \dfrac{1}{5}\displaystyle\int e^{(1+5\sin x)} 5\cos x\,dx.$



Making the substitution,

$$I = \frac{1}{5}\int e^u\,du$$
$$I = \frac{1}{5}e^u + c$$
$$I = \frac{1}{5}e^{(1+5\sin x)} + c.$$

18. Find $I = \int \frac{\sec^2(\ln x)}{x}dx$

a) $\sin(\ln x) + c$
b) $\tan(\ln x) + c$
c) $-\cos(\ln x) + c$
d) $-\cot(\ln x) + c$
e) None of the above.

Solution: b

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$.

$$I = \int \frac{\sec^2(\ln x)}{x}dx.$$

Making the substitution,

$$I = \int \sec^2 u\,du$$
$$I = \tan u + c$$
$$I = \tan(\ln x) + c.$$

19. Find $I = \int_{-1}^{\frac{\sqrt{3}}{3}-1} \frac{dx}{x^2+2x+2}$

a) $-\frac{\pi}{3}$ b) $-\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$ e) None of the above.

Solution: e

Let $u = x + 1 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$.

Note: $u = (-1) + 1 = 0$ and

$$u = (\frac{\sqrt{3}}{3} - 1) + 1 = \frac{\sqrt{3}}{3}.$$

$$I = \int_{-1}^{\frac{\sqrt{3}}{3}-1} \frac{dx}{x^2+2x+2} = \int_{-1}^{\frac{\sqrt{3}}{3}-1} \frac{dx}{x^2+2x+1+1}.$$
$$= \int_{-1}^{\frac{\sqrt{3}}{3}-1} \frac{dx}{(x+1)^2+1}$$

Making the substitution,

$$I = \int_0^{\frac{\sqrt{3}}{3}} \frac{du}{u^2+1}$$
$$I = \tan^{-1} u \Big|_0^{\frac{\sqrt{3}}{3}}$$
$$I = \tan^{-1}(\frac{\sqrt{3}}{3}) - \tan^{-1}(0)$$
$$I = \frac{\pi}{6} - 0$$
$$I = \frac{\pi}{6}.$$

20. In the first year in the college, students learn derivatives and Integral in Calculus I. These important tools have several applications in every branch of the physical sciences, actuarial science, computer science, statistics, engineering, economics, business, medicine, demography, etc. A problem is described by several variables that could be related by derivation or Integration. Given a function $f(x)$ or the graph of $f(x)$, what is the interpretation of its derivative (slope) or its Integral (Area)? In a discussion, three students said:

I. Student A: In a real life, only some variables are easy to measure. Then, you could study other variables by using Derivatives or Integrals.

II. Student B: I prefer to analyze the units of each variable of the problem to know the relation among them.

III. Student C: I understand that Derivatives mean division $\frac{dy}{dx}$ and Integrals mean product or area $\int f(x)dx$.

Then:

a) Only the student A 's idea is important and useful.
b) Only the student B 's idea is important and useful.
c) Only the student C 's idea is important and useful.

d) All Student's ideas could be combined to know when to applied Derivatives and Integrals to solve problems.
e) None of the above.

Solution: d

All Student's ideas could be combined to know when to apply Derivatives and Integrals to solve problems. Assume you are studying kinematic, the motion of points or bodies, by describing variables such as displacement $[m]$, speed $[m/s]$ and acceleration $[m/s^2]$. Student A is correct because acceleration is too difficult and expensive to be measured. Student B is correct because it is fundamental to analyze the units of each variable of the problem to know the relation among them. Student C is correct because Derivatives mean division $\frac{dy}{dx}$ and Integrals mean product or area $\int f(x)dx$. For example, the speed can be defined by the derivative of the displacement $v = \frac{ds}{dt}$ or the acceleration can be defined by the derivative of the speed $a = \frac{dv}{dt}$. The displacement can be defined by the Integral of the speed $\Delta s = \int v dt$ or the speed can be defined by the Integral of the acceleration $v = \int a dt$.

Name:________ **Age:**____ **Id:**________ **Course:**________

PART 2: SOLUTIONS

Consulting

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
3					
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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

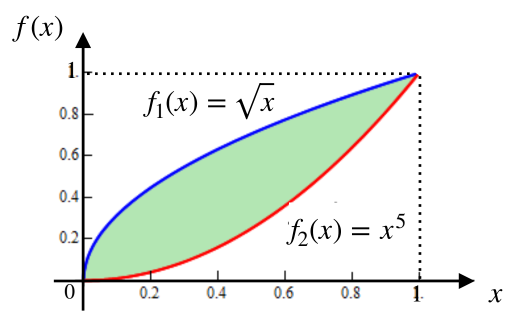
21. Show that $\int \sin x^3 \, dx = -\cos x + \frac{\cos^3 x}{3} + c.$

Hint: $\sin^2 x = 1 - \cos^2 x$

Solution:

$$I = \int \sin x^3 \, dx$$
$$I = \int \sin x^2 \sin x \, dx$$
$$I = \int (1 - \cos x^2) \ \sin x \, dx$$
$$I = \int \sin x \, dx - \int \cos x^2 \sin x \, dx$$
$$I = \int \sin x \, dx + \int \cos^2 x(-\sin x \, dx)$$
Let $u = \cos x \Rightarrow du = -\sin x \, dx$
$$I = \int \sin x \, dx + \int u^2 \, du$$
$$I = -\cos x + \frac{u^3}{3} + c$$
$$I = -\cos x + \frac{\cos^3 x}{3} + c$$
Thus, $\int \sin x^3 \, dx = -\cos x + \frac{\cos^3 x}{3} + c.$

22. Find the area between the curves $f_1(x) = \sqrt{x}$ and $f_2(x) = x^5$.



Solution:

$$A = \int_0^1 f_1(x) - f_2(x) \, dx$$
$$A = \int_0^1 \sqrt{x} - x^5 \, dx$$

$A = \int_0^1 x^{\frac{1}{2}}\,dx - x^5\,dx$

$A = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - \left[\frac{x^6}{6}\right]_0^1$

$A = \frac{2}{3} - \frac{1}{6}$

$A = \frac{1}{2}.$

23. Let $I_1 = \int_{-1}^{1} x^2(x)\,dx$ (using substitution) and $I_2 = \int_{-1}^{1} x^3\,dx$ (using power rule).

Show that $I_1 = I_2$

Solution:

Let $u = x^2 \Rightarrow du = 2dx.$
Note: $u = (-1)^2 = 1$ and $u = (1)^2 = 1$

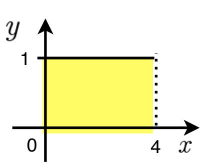
$I_1 = \int_{-1}^{1} x^2(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,2x\,dx$

Making a substitution,

$I_1 = \frac{1}{2}\int_1^1 u\,du = 0$

$I_2 = \int_{-1}^{1} x^3\,dx = \frac{u^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$

24. Find the area (A) using integrals.



Solution: $A = 4$

$A = \int_0^4 dx = x\Big|_0^4 = 4 - 0 = 4.$

*25. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(-x) = f(x)$ and $a \in \mathbb{R}$.

Show that $\int_0^a f(x)\,dx = A \Rightarrow \int_{-a}^{a} f(x)\,dx = 2A.$

Method 1: Graphically (Full Credit = 5 points).

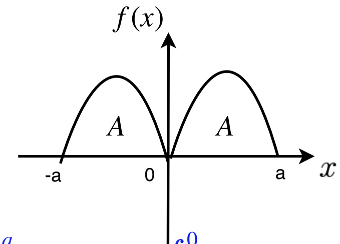
Method 2: Substitution (Full Credit = 5 points).

Note: Both method: (10 points)

Solution:

Method 1: Show graphically.

Since even function its graph is symmetric with respect to the origin and $f(-x) = f(x)$ then.



$I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$
$I = A + A = 2A.$

Method 2: Show by substitution.

(1) f is even $\Rightarrow f(-x) = f(x)$

(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Let $\int_0^a f(x)\,dx = A$

Let $t = -x$ by substitution the integral can be rewrite:

$A = \int_0^{-a} f(-t)\,(-dt) =$

$= -\int_0^{-a} f(t)\,dt \qquad (1)$

$= \int_{-a}^{0} f(t)\,dt \qquad (2)$

$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = 2A.$